Complementary two-dimensional finite element formulations with inclusion of a vectorized Jiles-Atherton model

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Abstract This paper deals with the magnetic vector and scalar potential formulation for two-dimensional (2D) finite element (FE) calculations including a vector hysteresis model, namely a vectorized Jiles-Atherton model. The particular case of a current-free FE model with imposed fluxes and magnetomotive forces is studied. The non-linear equations are solved by means of the Newton-Raphson method, which leads to the use of the differential reluctivity and permeability tensor. The proposed method is applied to a simple 2D model exhibiting rotational flux, viz the T-joint of a three-phase transformer.

Introduction
In the domain of numerical electromagnetism, the inclusion of hysteresis models in finite element (FE) field computations remains a challenging task (Chiampi et al., 1995; Dupré et al., 1998; Sadowski et al., 2002; Saitz, 1999). Mostly the scalar Preisach and Jiles-Atherton hysteresis models are used. They are applicable to 1D, 2D and 3D FE models displaying unidirectional flux (Chiampi, 1995; Sadowski et al., 2002; Saitz, 1999). In applications having rotational flux in part of the computation domain, a vector hysteresis model should be used (Dupré et al., 1998). The non-linear equations are iteratively solved by means of the fixed-point method (Chiampi et al., 1995; Saitz, 1999) or the Newton-Raphson method (Dupré et al., 1998; Sadowski et al., 2002; Saitz, 1999). The latter method has the advantage of fast convergence (near the exact solution), but is somewhat more complicated to implement.

For 2D magnetic field computations, including those with hysteresis, the vector potential formulation is almost invariably adopted: the vector potential has only one non-zero component (along the third dimension) and the formulation is very easy to implement. The scalar potential formulation is rarely used as it requires the calculation...
of sources fields and the definition of cuts, unless the (2D or 3D) domain is current-free and simply-connected. However, the incorporation of a vector hysteresis model is very analogous in both formulations, as will be shown in this paper. Applying the Newton-Raphson method in a somewhat uncommon way, the differential reluctivity and permeability tensors, respectively, naturally emerge (Dupré et al., 1998).

In this paper, we will consider a current-free 2D FE model with imposed fluxed and magnetomotive forces (Dupré et al., 1998; Dular et al., 1999) and a vector generalization of the Jiles-Atherton model (Bergqvist, 1996). The duality of the two formulations will be pointed out and some results for a simple model with rotational flux will be presented.

**Complementary formulations**

**Governing equations**

We consider a simply-connected and current-free domain $\Omega$ in the $xy$-plane. The magnetic field vector $h(x,y)$ and the induction vector $b(x,y)$ both have a zero $z$-component and are related by the magnetic constitutive law $b = b(h)$ or $h = h(b)$.

For any continuous one-component vector potential $\varphi = \varphi(x,y)$ and scalar potential $u(x,y)$, the induction $b = \text{curl} \ \varphi$ and the magnetic field $h = -\text{grad} u$ automatically satisfy $\text{div} \ h = 0$ and $\text{curl} \ h = j = 0$, respectively. The FE discretisation of $\Omega$ leads to the definition of basis functions $\varphi_l(x,y) = \alpha_l(x,y)|_\Omega$ and $u_l(x,y)$ for the potentials $\varphi$ and $u$:

$$\varphi(x,y,t) = \sum_{l=1}^{n} a_l \varphi_l(x,y)$$

$$u(x,y,t) = \sum_{l=1}^{n} u_l \alpha_l(x,y)$$

Commonly triangular elements and piecewise linear nodal basis functions are adopted.

The weak form of Ampère’s law $\text{curl} \ h = j = 0$ and the flux conservation law $\text{div} \ b = 0$ reads, before and after partial integration:

$$\left(\text{curl} \ h, \alpha_k^l\right)_\Omega = 0 \Rightarrow (h, \text{curl} \ \alpha_k^l)_\Omega + \langle h \times n, \alpha_k^l \rangle_\Gamma = 0,$$  \hspace{1cm} (3)

$$\left(\text{div} \ b, \alpha_k^l\right)_\Omega = 0 \Rightarrow (b, \text{grad} \ \alpha_k^l)_\Omega + \langle b \cdot n, \alpha_k^l \rangle_\Gamma = 0,$$  \hspace{1cm} (4)

where $\alpha^l_k = \alpha^l_k(x,y)|_\Omega$ and $\alpha^l_k(x,y)$ are continuous test functions; $(\cdot, \cdot)_\Omega$ and $(\cdot, \cdot)_\Gamma$ denote the integral of the (scalar) product of the two vector or scalar arguments over the domain $\Omega$ and on its contour $\Gamma$, respectively; $n$ is the inward unit normal on $\Gamma$.

Considering the basis functions as test functions, a system of algebraic equations is obtained. For linear isotropic materials, having a constant scalar reluctivity $\nu$ and permeability $\mu = \nu^{-1}$, with $b = \mu h$, the elements of the system matrices (that are not affected by the boundary conditions) are given by the following expressions:

$$\left(\nu \text{curl} \ \alpha_l, \text{curl} \ \alpha_k\right)_\Omega = (\nu \text{grad} \ \alpha_l, \text{grad} \ \alpha_k)_\Omega,$$  \hspace{1cm} (5)

$$\left(\mu \text{grad} \ \alpha_l, \text{grad} \ \alpha_k\right)_\Omega.$$  \hspace{1cm} (6)

**Boundary conditions with flux walls and flux gates**

Let us consider the case where the boundary $\Gamma$ is a sequence of the so-called flux walls $\Gamma_{\text{wi}}$ and flux gates $\Gamma_{\text{gi}}$ (Dupré et al., 1998; Dular et al., 1999). A flux wall $\Gamma_{\text{wi}}$ is an interface with an impermeable medium ($\mu = 0 \Rightarrow b = 0$), on which thus holds $b \cdot n = 0$; the
associated magnetomotive force is \( F_i = \langle h \times n, 1 \rangle_{\Gamma_{\text{w}i}} \). A flux gate \( \Gamma_{\text{g}i} \) is an interface with a perfectly permeable medium (\( \mu = \infty \Rightarrow h = 0 \)), on which thus holds \( h \times n = 0 \); the flux through the gate, inward \( \Omega \), is given by \( \Phi_i = \langle h \cdot n, 1 \rangle_{\Gamma_{\text{g}i}} \). It follows that the sum of the magnetomotive forces \( F_i \) is zero, as well as the sum of the fluxes \( \Phi_i \).

An example with three flux walls and three flux gates is shown in Figure 1.

In the \( a \)-formulation, \( a(x, y) \) has a constant value \( A_{\text{wi}} \) on each flux wall \( \Gamma_{\text{w}i} \). Gate fluxes \( \Phi_k = A_{\text{wk}} - A_{\text{w}l} \), or linear combinations of gate fluxes, can be strongly imposed by fixing two or more \( A_{\text{wi}} \) values. (At least one \( A_{\text{wi}} \) is to be set, e.g. to zero, in order to ensure the uniqueness of \( a \).) An \( A_{\text{wi}} \) value can also constitute an unknown of the problem; this is a so-called floating potential. The corresponding magnetomotive force \( F_i \) is then given, and weakly imposed via the contour integral in equation (3). Hereto a dedicated basis function, denoted by \( \alpha_{\text{wi}}(x, y) = \alpha_{\text{wi}}(x, y) \downarrow \), is defined. It has value 1 on \( \Gamma_{\text{wi}} \) and decreases linearly to 0 in the layer of elements surrounding \( \Gamma_{\text{wi}} \); it is the sum of the classical nodal basis functions \( a_i \) associated with the nodes situated on \( \Gamma_{\text{wi}} \).

Analogously, in the \( u \)-formulation, flux gates have \emph{a priori} known or floating potential \( u(x, y) = U_{\text{gi}} \). In the latter case, the flux through the gate \( \Phi_i \) is weakly imposed via the contour integral in equation (4).

**Linear test case**

By way of example, some results for a linear magnetostatic case (T-joint of Figure 1, \( \Phi_1 = \Phi_2 = 1, \nu = \mu = 1 \)) are shown in Figures 2-4. The magnetomotive forces obtained with the two formulations (Figure 3) are observed to converge monotonously to each other, which is certainly not the case for the local induction value considered (Figure 4).
Newton-Raphson method

Let us first consider a reversible non-linear isotropic material in \( \Omega \); hysteretic media will be dealt with in the next section. The scalar reluctivity and permeability can be written as a single-valued function of (the square of) the magnitude of \( b \) and \( h \):

\[
\nu = \nu(b^2) \quad \text{and} \quad \mu = \mu(h^2).
\]

The systems of algebraic equations are non-linear and have to be solved iteratively. The Newton-Raphson method is commonly used as it offers quadratic convergence near the exact solution. Starting from initial (zero) solutions \( a(0) \) and \( u(0) \), subsequent approximations \( a(i) = a(i-1) + \Delta a(i) \) and \( u(i) = u(i-1) + \Delta u(i), i = 1, 2, \ldots \), are obtained by linearising the non-linear systems around the \((i-1)\)th solutions \( a(i-1) \) and \( u(i-1) \). The linearization of equations (3) and (4) requires the evaluation of their derivatives with respect to degrees of freedom \( a_l \) and \( u_l \). Given that

\[
\frac{\partial h}{\partial a_l} = \frac{\partial h}{\partial b} \text{curl } a_l \quad \text{and} \quad \frac{\partial b}{\partial u_l} = - \frac{\partial b}{\partial h} \text{grad } a_l,
\]

Figure 3.
Magnetomotive force \( \Phi_2 = -\Phi_3 \) as a function of number of degrees of freedom

Figure 4.
Magnitude of \( b \) in point \( p \) as a function of number of degrees of freedom
the elements of the matrix of the linearised systems can be concisely written in terms of the differential reluctivity and permeability tensors \( \frac{\partial h}{\partial b} \) and \( \frac{\partial b}{\partial h} \):

\[
\left( \frac{\partial h}{\partial b} \right)_{\text{curl} \alpha_l, \text{curl} \alpha_k} \quad \text{and} \quad \left( \frac{\partial b}{\partial h} \right)_{\text{grad} \alpha_l, \text{grad} \alpha_k}.
\] (8)

For the isotropic materials considered, these tensors can be expressed in terms of the functions \( \nu = \nu(b^2) \) and \( \mu = \mu(h^2) \) and their derivatives:

\[
\frac{\partial h}{\partial b} = \nu_1 + 2 \frac{d\nu}{db^2} bb \quad \text{and} \quad \frac{\partial b}{\partial h} = \mu_1 + 2 \frac{d\mu}{dh^2} bb,
\] (9)

where \( bb \) and \( hh \) are the dyadic squares of \( b \) and \( h \), and \( \mathbf{1} \) is the unit tensor. In the \( xy \) coordinate system, the matrix representation of, e.g. the reluctivity tensor is

\[
\begin{bmatrix}
\frac{\partial h}{\partial b} \\
\frac{\partial h}{\partial b}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial h_x}{\partial b_x} & \frac{\partial h_x}{\partial b_y} \\
\frac{\partial h_y}{\partial b_x} & \frac{\partial h_y}{\partial b_y}
\end{bmatrix} = \nu \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} + 2 \frac{d\nu}{db^2} \begin{bmatrix}
b_x b_x & b_x b_y \\
b_y b_x & b_y b_y
\end{bmatrix}.
\] (10)

It follows that the expression in equation (8) for the elements of the Jacobian matrices are equivalent with the more classical ones:

\[
\begin{align*}
\nu \text{curl} \alpha_l \cdot \text{curl} \alpha_k + 2 \frac{d\nu}{db^2} (\text{curl} \alpha_l \cdot b)(\text{curl} \alpha_k \cdot b) &= \\
\nu \text{grad} \alpha_l \cdot \text{grad} \alpha_k + 2 \frac{d\nu}{db^2} (\text{grad} \alpha_l \cdot \text{grad} a)(\text{grad} \alpha_k \cdot \text{grad} a)
\end{align*}
\] (12)

and

\[
\begin{align*}
\mu \text{grad} \alpha_l \cdot \text{grad} \alpha_k + 2 \frac{d\mu}{dh^2} (\text{grad} \alpha_l \cdot \text{grad} u)(\text{grad} \alpha_k \cdot \text{grad} u),
\end{align*}
\] (13)

which are obtained when deriving the non-linear equations on the basis of equations (5) and (6).

At the \( i \)th Newton-Raphson iteration, the Jacobian matrices and in particular the differential reluctivity and permeability tensors are evaluated for \( b = b_{(i-1)} \) and \( h = h_{(i-1)} \), respectively. The right hand side vector is composed of the residuals \( (h(b_{(i-1)}) \), \( \text{curl} \alpha_k)_{(i)} \) and \( (b(h_{(i-1)}) \), \( \text{grad} \alpha_k)_{(i)} \), respectively, where for the sake of brevity the contour terms and associated boundary conditions have been omitted. Resolution of the linearised systems produces the increments \( \Delta g_{(i)} \) and \( \Delta u_{(i)} \), and the \( i \)th solutions \( g_{(i)} \) and \( u_{(i)} \).

**Jiles-Atherton model**

**Scalar model**

In the scalar Jiles-Atherton model (Bergqvist, 1996; Chiampi et al., 1995; Sadowski et al., 2002), the material is characterized by five (scalar) parameters \( \alpha, a, m_s, c \) and \( k \).
The equations relevant to its vectorization (Bergqvist, 1996) and the FE implementation are briefly given hereafter.

The scalar magnetisation $m = b/\mu_0 - h$ is the sum of a reversible part $m_r$ and an irreversible part $m_i$, with

$$m_i = (m - cm_an)/(1 - c)$$  \hspace{1cm} (14)

$$m_r = c(m_an - m_i),$$  \hspace{1cm} (15)

where the anhysteretic magnetization $m_an$ is a single-valued function of the effective field $h_e = h + am$:

$$m_an(h_e) = m_s \left( \coth \left( \frac{h_e}{a} \right) - \frac{a}{h_e} \right).$$  \hspace{1cm} (16)

The irreversibility of the material is contained in

$$\frac{dm_i}{dh_e} = \frac{1}{\delta}(m_an - m_i) \quad \text{with} \quad \delta = \text{sign} \left( \frac{dh}{dt} \right).$$  \hspace{1cm} (17)

An alternative definition may be adopted in order to prevent $dm_i/dh_e$ and $db/dh$ from becoming negative (Bergqvist, 1996):

$$\frac{dm_i}{dh_e} = \frac{|m_an - m_i|}{k} \quad \text{if} \quad dh \cdot (m_an - m_i) > 0, \quad \text{else} \quad \frac{dm_i}{dh_e} = 0.$$  \hspace{1cm} (18)

The differential susceptibility $dm/db$ and the differential permeability $db/dh$ can then be calculated for the given $b$, $h$ and sign(dh):

$$\frac{db}{dh} = \mu_0 \left( 1 + \frac{dm}{dh} \right) \quad \text{and} \quad \frac{dm}{dh} = \frac{c \frac{dm_an}{dh_e} + (1 - c) \frac{dm_i}{dh_e}}{1 - ac \frac{dm_an}{dh_e} - a(1 - c) \frac{dm_i}{dh_e}}.$$  \hspace{1cm} (19)

For a given state $(h_1, b_1)$ at an instant $t_1$, $h_2$ at a later instant $t_2$ can be calculated when given $b_2$, and vice versa:

$$b_2 = b_1 + \int_{h_1}^{h_2} \frac{db}{dh} dh \quad \text{and} \quad h_2 = h_1 + \int_{b_1}^{b_2} \frac{dh}{db} db,$$  \hspace{1cm} (20)

where $dh/db$ is the inverse of $db/dh$. The integration has to be carried out numerically. Unfortunately, the integrand does not only depend on the integration variable, i.e. $h$ and $b$, respectively, but also on $b$ and $h$, respectively. Therefore, a Gauss integration cannot be applied (as such).

**Vector extension**

We now outline the vector extension as proposed by Bergqvist (1996), but limit the analysis to the isotropic case. In the vector generalization of equations (14-16)
and (18-20), the scalar fields are replaced by vector fields, e.g. \( b \) becomes \( \mathbf{b} \), while the scalar differential quantities are replaced by tensors, e.g. \( d\mathbf{b}/dh \) becomes \( \partial\mathbf{b}/\partial h \). The division in equation (19) is replaced by the multiplication of the nominator by the inverse of the denominator. The scalar 1 is replaced by the unit tensor \( I \) where necessary. The vector extension of equations (16) and (18) needs special attention.

\[
m_{an} \text{ and } \partial m_{an}/\partial h_e \text{ are single-valued functions of } h_e:
\]

\[
m_{an} = m_{an}(h_e) \frac{h_e}{h_e^2},
\]

\[
\frac{\partial m_{an}}{\partial h_e} = m_{an} \left( 1 - \frac{h_e h_e}{h_e^2} \right) + \frac{dm_{an}}{dh_e} \frac{h_e h_e}{h_e^2}.
\]

According to Bergqvist (1996), the vector extension of equation (18) consists in assuming that the increment \( dm_i \) is parallel to \( m_{an} - m_i \), proportional to \( (m_{an} - m_i)/k \) and non-zero only if \( dh \cdot (m_{an} - m_i) > 0 \). Considering a local coordinate system \( x'y' \), with the \( x' \)-axis along the vector \( m_{an} - m_i \) (Figure 5), we thus have

\[
\begin{bmatrix}
\frac{\partial m_i}{\partial h_e}
\end{bmatrix}_{x'y'} = \frac{|m_{an} - m_i|}{k} \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \text{ if } dh \cdot (m_{an} - m_i) > 0, \text{ else } \frac{\partial m_i}{\partial h_e} = 0.
\]

The matrix representation of \( \partial m_i/\partial h_e \) in a coordinate system \( xy \) is then

\[
\begin{bmatrix}
\frac{\partial m_i}{\partial h_e}
\end{bmatrix}_{xy} = R \begin{bmatrix}
\frac{\partial m_i}{\partial h_e}
\end{bmatrix}_{x'y'} R^T
\]

with

\[
R = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}.
\]

Using all the above equations (or their vector extension), \( \partial \mathbf{b}/\partial h \) can be calculated for the given \( \mathbf{b} \) and \( h_e \) and given direction of \( dh \). By inverting (the matrix representation of) \( \partial \mathbf{b}/\partial h, \partial \mathbf{h}/\partial \mathbf{b} \) is obtained.

Some calculated \( bh \)-loci (with \( \mu_0 ms = 2.1 \text{T}, a = 50 \text{A/m}, k = 82 \text{A/m}, c = 0.1 \) and \( \alpha = k/ms \) (Bergqvist, 1996)) are shown in Figure 6. Both alternating and rotational excitations are considered.

Figure 5. Local coordinate system \( x'y' \) with \( x' \)-axis along \( m_{an} - m_i \)
Incorporation in FE equations

For hysteretic material models, the differential reluctivity and permeability tensors depend on the present state \((b, h)\) of the material as well as on the history of the material. For the vector Preisach model considered in (Dupré et al., 1998), the history consists of extreme values of the magnetic field projected on a number of spatial directions. In the above outlined vectorized Jiles-Atherton model, the history is simply contained in the direction of \(dh\).

For stepping from the instant \(t_1\) to the next instant \(t_2 = t_1 + \Delta t\), the \(i\)th Newton-Raphson iteration requires the evaluation of the differential tensors \(\partial h / \partial b\) and \(\partial b / \partial h\) for \(b = b_{2(i-1)}, h = h_{2(i-1)}\) and \(dh = h_{2(i)} - h_{2(i-1)}\). After solving the systems of equations in terms of \(\Delta a_{b(i)}\) and \(\Delta u_{b(i)}\), \(\bar{b}_{2(i)}\) and \(\bar{b}_{20}\) are obtained by integrating the differential tensors over \([\bar{b}_1, \bar{b}_{2(i)}]\) and \([\bar{b}_1, \bar{b}_{20}]\), respectively.

Application example

The vector Jiles-Atherton model (with the parameter values given above) is applied to the T-joint model considered in the complementary formulations section. The fluxes \(\Phi_1 = \cos(2\pi ft + 2\pi/3)\) and \(\Phi_2 = \cos(2\pi ft)\), where the frequency \(f\) is arbitrarily chosen to be 1 Hz, are imposed strongly in the \(a\)-formulation and weakly in the \(u\)-formulation. Two periods are time-stepped with 200 time steps per period. During the first quarter of a period, the fluxes \(\Phi_1\) and \(\Phi_2\) are multiplied with the function \((1 - \cos(\pi t/t_{\text{relax}})))/2\), with \(t_{\text{relax}} = 0.25\), in order to step smoothly through the first magnetization curve of the hysteretic material. The mesh with 661 spatial degrees for \(a(x, y, t)\) and 715 for \(u(x, y, t)\) is used.

The magnetomotive forces \(\mathcal{F}_1(t)\) and \(\mathcal{F}_2(t)\) obtained with \(a\)- and \(u\)-formulations are shown in Figure 7. A very good agreement is reached. The \(b\)-locus and \(b_x h_x\) and \(b_y h_y\)-loops in the point \(p\) (shown in Figure 2) are shown in Figure 8. The agreement is somewhat less good, as could be expected for a local quantity.

Conclusions

The implementation of a vectorized Jiles-Atherton model in 2D FE magnetic field computations with complementary formulations has been studied. When solving the non-linear equations by means of the Newton-Raphson method, the differential reluctivity and differential tensors naturally emerge.

The proposed methods have been successfully applied to a simple 2D FE model with rotational flux. A good agreement has been achieved between the results obtained with the two formulations.

Figure 6.

\(bh\)-loci at alternating excitation (left) and \(b_x h_x\)-loci (or \(b_y h_y\)-loci) at rotational excitation (right), with \(h = 100, 150\) and 300 A/m
References


